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THE LONG-TERM MEMORY EFFECT IN HOMOGENEOUS PLATES†

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It is shown that the long-term memory effect, observed in composite materials [1, 2], can appear in plates and in this case it need not be associated with existence of an inhomogeneous structure. An analysis of the problem is carried out using a two-scale asymptotic expansion [3].

1. FORMULATION OF THE PROBLEM

LET US consider the three-dimensional problem in a thin layer $S\times[-\epsilon/2, \epsilon/2]$ of constant thickness ϵ (as $\epsilon\to 0$, the domain contracts to the two-dimensional area $S\subset R^2$ which is a plate). We have

$$S_{ij,j}=f_i \text{ in } S\times[-\epsilon/2,\epsilon/2] \tag{1.1}$$

$$\mathbf{u}(\mathbf{x},t)=0 \text{ on } \partial S\times[-\epsilon/2,\epsilon/2] \tag{1.2}$$

$$\sigma_{i3}=g_i^{\pm} \text{ on surfaces } \{\mathbf{x}'\in S,\ x_3=\pm\epsilon/2\}$$

$$\mathbf{u}(\mathbf{x},0)=0 \text{ in } S\times[-\epsilon/2,\epsilon/2] \tag{1.3}$$

Here $\{\sigma_{ij}\}$ is the stress tensor, $\mathbf{u}$ is the displacement vector of the plate, regarded as a three-dimensional body, and the components of vector $\mathbf{x}'=(x_1,x_2)$ are coordinates in the plane of the plate.

We take the governing relations in the form

$$\sigma_{ij}=\epsilon^a(a_{ijkl}u_{k,l}+\Gamma_{ijkl}u_{k,l}) \tag{1.4}$$

where $\{a_{ijkl}\}$ is the tensor of the elastic constants, $\{\Gamma_{ijkl}\}$ is the tensor of linear operators (i.e. linear with respect to time). For the case of a visco-elastic material we have $\Gamma_{ijkl}=b_{ijkl}\partial/\partial t$. The multiplier ϵ^a of (1.4), when $a=-1$, ensures zero stiffness of the plate in its plane and, when $a=-3$, it ensures non-zero flexural stiffness as $\epsilon\to 0$.

We will consider the most general case when $a=-3$ (which also includes, as we shall see, the case $a=-1$). The quantities a_{ijkl}, Γ_{ijkl} (like b_{ijkl}, if the material is visco-elastic) are assumed to be bounded uniformly with respect to ϵ.

2. ASYMPTOTIC EXPANSION

Consider a plate of constant thickness which is fabricated from a homogeneous material. We will use the following asymptotic expansion (which is a special case of the expansion given earlier in [3])

$$\begin{gathered}\mathbf{u}=\mathbf{u}^{(0)}(\mathbf{x}',t)+\epsilon\mathbf{u}^{(1)}(\mathbf{x}',y_3,t)+\dots\\ \sigma_{ij}=\epsilon^{-3}\sigma_{ij}^{(-3)}(\mathbf{x}',y_3,t)+\epsilon^{-2}\sigma_{ij}^{(-2)}(\mathbf{x}',y_3,t)+\dots;\quad y_3=x_3/\epsilon\end{gathered} \tag{2.1}$$

Substituting expansion (2.1) into (1.1), substituting $\epsilon^{-1}\partial/\partial y_3$ and $\partial/\partial x_\alpha$ for the differentiation operators $\partial/\partial x_i$ (henceforth Greek indices take the values 1, 2, and Latin ones take the values 1, 2, 3), and equating expressions of like powers of ϵ, we obtain

$$\begin{gathered}\sigma_{i3,3y}^{(m+1)}+\sigma_{i\alpha,\alpha x}^{(m)}=0,\quad m\neq 0\\ ,3y=\partial/\partial y_3,\quad ,\alpha x=\partial/\partial x_\alpha\end{gathered} \tag{2.2}$$

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Averaging Eqs (2.2) over the plate thickness and averaging the same equations, first multiplied by y_3, we obtain the equations of equilibrium

$$\langle \sigma_{\alpha\beta}^{(-3)} \rangle_{,\alpha x} = 0, \quad \langle \sigma_{\beta\alpha}^{(-2)} \rangle_{,\alpha x} + g_\beta^+ + g_\beta^- = 0$$
$$\langle \sigma_{3\alpha}^{(-1)} \rangle_{,\alpha x} + g_3^+ + g_3^- = 0, \quad M_{\beta\alpha,\alpha x}^{(-2)} - \langle \sigma_{\beta 3}^{(-1)} \rangle + g_\beta^+ + g_\beta^- = 0 \tag{2.3}$$

Here $\langle \cdot \rangle = \int_{-1/2}^{1/2} \cdot \, dy_3$ is the mean value over the plate thickness, $\langle \sigma_{ij}^{(m)} \rangle$ are forces, and $M_{ij}^{(m)} = \langle y_3 \sigma_{ij}^{(m)} \rangle$ are the moments of the forces.

For the plate, but not for the material from which it is fabricated, the governing equations must relate the forces and the moments of the forces to the deformation characteristics (when the asymptotic method is used they turn out to be the classical deformations in the plane of the plate and the curvatures).

Substituting expansion (2.1) into the governing equations (2.2) of the material of the plate, replacing the differentiation operators and equating the expressions of like power of ϵ, we find

$$\sigma_{ij}^{(m)} = a_{ijk\alpha} u_{k,\alpha x}^{(m+3)} + a_{ijk3} u_{k,3y}^{(m+4)} + \Gamma_{ijk\alpha} u_{k,\alpha x}^{(m+3)} + \Gamma_{ijk3} u_{k,3y}^{(m+4)} \tag{2.4}$$

When $m = -3$, substitution of expressions (2.4) into (2.2) leads to the problem (y_3 and t are variables, $\mathbf{x}'$ is a parameter, and $\gamma_{\alpha\beta}$ are deformations in the plane of the plate)

$$\sigma_{i3,3y}^{(-3)} = 0, \quad \sigma_{i3}^{(-3)} = 0 \text{ when } y_3 = \pm 1/2 \tag{2.5}$$

Here

$$\sigma_{ij}^{(-3)} = a_{ijk3} u_{k,3y}^{(1)} + \Gamma_{ijk3} u_{k,3y}^{(1)} + a_{ij\beta\alpha}\gamma_{\beta\alpha} + \Gamma_{ij\beta\alpha}\gamma_{\beta\alpha} + a_{ij3\alpha} u_{3,\alpha x}^{(0)} + \Gamma_{ij3\alpha} u_{3,\alpha x}^{(0)} \tag{2.6}$$

Substituting (2.1) into (1.3) we obtain the intial conditions

$$\mathbf{u}^{(k)}(\mathbf{x}', y_3, 0) = 0, \quad k = 0, 1, 2, \ldots \tag{2.7}$$

and, in particular,

$$\mathbf{u}^{(1)}(\mathbf{x}', y_3, 0) = 0 \tag{2.8}$$

The solution of problem (2.5), (2.6) and (2.8) gives an expression for $\mathbf{u}^{(1)}$ in terms of $\{u_{3,\alpha x}^{(0)}\}$, $\{\gamma_{\beta\alpha}\}$. Substituting this expression into (2.6) and integrating the result over the plate thickness, we find the relation between the forces and $\{\gamma_{\beta\alpha}\}$ (as we will see, $\{u_{3,\alpha x}^{(0)}\}$ do not occur in this relation).

3. ANALYSIS OF THE GOVERNING RELATIONS. EXTENSION IN THE PLANE OF THE PLATE

Let us represent the solution of problem (2.5), (2.6) and (2.8) as a sum of terms corresponding to deformations of the plate in its plane and to its bending, namely $\mathbf{u}^{(1)} = \mathbf{u}_1^{(1)} + \mathbf{u}_2^{(1)}$ where $\mathbf{u}_1^{(1)}$ is the solution of the problem

$$\sigma_{i3,3y}^{u} = 0, \quad \sigma_{i3}^{u} = 0 \text{ when } y_3 = \pm 1/2 \tag{3.1}$$

where

$$\sigma_{i3}^{u} = a_{i3k3} u_{1k,3y}^{(1)} + \Gamma_{i3k3} u_{1k,3y}^{(1)} + a_{i3\beta\alpha}\gamma_{\beta\alpha} + \Gamma_{i3\beta\alpha}\gamma_{\beta\alpha} \tag{3.2}$$

We will seek the solution of (3.1), (3.2) in the form (bearing in mind that the quantities occurring in (3.1) and (3.2) are independent of y_3)

$$\frac{\partial}{\partial y_3} u_{1k}^{(1)}(x', y_3, t) = V_k(t), \text{ i.e. } u_{1k}^{(1)}(x', y_3, t) = V_k(t) y_3 + C_k \tag{3.3}$$

Substituting (3.3) into (3.2) and taking into account the boundary conditions we obtain

$$\sigma_{i3}^{u} = a_{i3k3} V_k + \Gamma_{i3k3} V_k + a_{i3\beta\alpha}\gamma_{\beta\alpha} + \Gamma_{i3\beta\alpha}\gamma_{\beta\alpha} =$$
$$= (a_{i3k3}) \{ \mathbf{V} + (a_{i3k3})^{-1} (a_{i3\beta\alpha}\gamma_{\beta\alpha}) \} + (\Gamma_{i3k_3}) \{ \mathbf{V} + (\Gamma_{i3k3})^{-1} (\Gamma_{i3\beta\alpha}) \gamma_{\beta\alpha}) \} = 0 \tag{3.4}$$

where (a_{i3k3}) and (Γ_{i3k3}) are 3×3 matrices (numerical and operator matrices), -1 indicates their inversion (in the corresponding sense), $\mathbf{V} = (V_1, V_2, V_3)$ and $(a_{i3\beta\alpha}\gamma_{\beta\alpha})_{i=1,2,3}$ and $(\Gamma_{i3\beta\alpha}\gamma_{\beta\alpha})_{i=1,2,3}$ are vectors.

In the general case (since Γ_{ijkl} are time-operators) (3.1) is an operator equation (in the case of a visco-elastic material it is a differential equation) and its solution is given by a resolving operator that is non-linear with respect to time.

Remark. In the special case, when the relations

$$(a_{i3k3})^{-1}(a_{i3\beta\alpha}\gamma_{\beta\alpha})_{i=1,2,3} = (\Gamma_{i3k3})^{-1}(\Gamma_{i3\beta\alpha}\gamma_{\beta\alpha})_{i=1,2,3} \tag{3.5}$$

hold (cf. [2]), equality (3.4) is satisfied by the function

$$\mathbf{V} = -(a_{i3k3})^{-1}(a_{i3\beta\alpha}\gamma_{\beta\alpha})_{i=1,2,3} \tag{3.6}$$

Consider the function $u_2^{(1)}$ contained in the equality $\mathbf{u}^{(1)} = \mathbf{u}_1^{(1)} + \mathbf{u}_2^{(1)}$. It is determined by solving problem (3.1) with

$$\sigma_{i3}^{u} = a_{i3k3}u_{k,3y}^{(1)} + \Gamma_{i3k3}u_{k,3y}^{(1)} + a_{i33\alpha}u_{3,\alpha x}^{(0)} + \Gamma_{i33\alpha}u_{3,\alpha x}^{(0)}$$

The solution of this problem, as has been stated in [2], is the function $u_k^{(1)} = -y_3\delta_{k\alpha}u_{3,\alpha x}^{(0)}$. This can be verified by substitution (taking into account the symmetry of the coefficients $a_{i33\alpha} = a_{i3\alpha 3}$, $\Gamma_{i33\alpha} = \Gamma_{i3\alpha 3}$).

For $m = -3$, substitution of the expressions obtained into (2.4), after integrating the result over the plate thickness, yields

$$\langle \sigma_{ij}^{(-3)} \rangle = \langle a_{ijk3}V_k + \Gamma_{ijk3}V_k + a_{ij\beta\alpha}\gamma_{\beta\alpha} + \Gamma_{ij\beta\alpha}\gamma_{\beta\alpha} \rangle \tag{3.7}$$

We see that Eqs (3.7), which are the governing equations of the plate, are of the same type as the governing equations (2.1) of the material if (3.5) is satisfied. This does not hold in the general case.

Example. Consider a plate fabricated from a visco-elastic material characterized by the equations $\Gamma_{ijkl} = b_{ijkl}\partial/\partial t$, $b_{ijkl} = \text{const}$. For simplicity, we consider the case of uniaxial extension: $\gamma_{\beta\alpha} = \gamma_{22}\delta_{\beta 2}\delta_{\alpha 2}$. In this case (3.4) takes the form (taking into account that for the case of an isotropic material, which is considered here, (a_{i3k3}) and (Γ_{i3k3}) are diagonal matrices [4])

$$\sigma_{i3}^{u} = a_{i3i3}V_i + a_{i322}\gamma_{22} + b_{i3i3}\frac{\partial V_i}{\partial t} + b_{i322}\frac{\partial\gamma_{22}}{\partial t} =$$

$$= a_{i3i3}\Big(V_i + \frac{a_{i233}}{a_{i3i3}}\gamma_{22}\Big) + b_{i3i3}\frac{\partial}{\partial t}\Big(V_i + \frac{b_{i233}}{b_{i3i3}}\gamma_{22}\Big) \tag{3.8}$$

For isotropic materials, taking account of the relations $a_{i322} = 0$ and $b_{i322} = 0$ for $i = 1, 2$, (3.8) takes the form (for $i = \alpha = 1, 2$)

$$\sigma_{\alpha 3}^{u} = a_{\alpha 3\alpha 3}V_\alpha + \frac{\partial}{\partial t}b_{\alpha 3\alpha 3}V_\alpha$$

and from (3.3) it follows that $V_\alpha = 0$ ($\alpha = 1, 2$). In the case being considered only the third equation ($i = 3$) of (3.1) thus remains. It reduces to [cf. (3.4)]

$$a_{3333}(V_3 + A\gamma_{22}) + b_{3333}\partial(V_3 + A\gamma_{22})/\partial t = -b_{3333}\partial(B - A)/\partial t$$
$$A = a_{3322}/a_{3333}, \quad B = b_{3322}/b_{3333} \tag{3.9}$$

if relations (3.8) are taken into account.

By (3.3) and (2.8) we have

$$V_3(0) = 0 \tag{3.10}$$

(recall that V_3 is a function of the single argument t). The solution of Eqs (3.9) and (3.10) is

$$V_3(t) = A(-\gamma_{22}(t) + \gamma_{22}(0)) + \int_0^t e^{C\tau}\Delta\frac{\partial\gamma_{22}}{\partial t}(\tau)\,d\tau \quad \Big(C = \frac{a_{3333}}{b_{3333}}\Big), \quad \Delta = B - A \tag{3.11}$$

Substitution of expression (3.11) into (3.7) leads to the following governing relations (for the case considered, when only the quantity γ_{22} is non-zero)

$$\langle \sigma_{ij}^{(-3)} \rangle = a_{ij33}A(-\gamma_{22}(t) + \gamma_{22}(0)) + a_{ij33}\Delta\int_0^t e^{C\tau}\frac{\partial\gamma_{22}}{\partial t}(\tau)\,d\tau +$$

$$+ b_{ij33}\Big(-A\frac{\partial\gamma_{22}}{\partial t}(t) + \Delta e^{Ct}\frac{\partial\gamma_{22}}{\partial t}(t)\Big) + a_{ij22}\gamma_{22}(t) + b_{ij22}\frac{\partial\gamma_{22}}{\partial t}(t) \tag{3.12}$$

On the right-hand side of (3.12) the angle bracket symbols are omitted since the functions contained within them do not depend on y_3 and for them the mean of a function equals the function itself.

As is seen, the governing equations of the plate fabricated from a visco-elastic material (which is described by the local, with respect to time, governing relations) contain the non-local integral term (which is of the kind considered in the theory of hereditary elasticity [4]) and the term dependent on time explicitly (which is of the kind considered in the theory of ageing [4]). Let us estimate the magnitude of these terms. The integral term is of the order of (the integral of (3.12) is evaluated at $\partial\gamma_{22}/\partial t = 1$)

$$a_{ij33}/C^{-1}(B-A)(e^{C\tau}-1)$$

The interval of significant fading by a factor of e of the memory equals $|1/C|$ for $C<0$. For $C>0$ the plate has a non-decaying memory. The quantities A and B are of the order of the elastic and "viscous" Poisson's ratios, while $1/C$ is of the order of the ratio of the coefficient of viscosity of the material to its Young's modulus.

4. ANALYSIS OF THE GOVERNING RELATIONS. BENDING

By virtue of the above considerations we have

$$u_k^{(1)} = u_{1k}^{(1)} + u_{2k}^{(1)} = u_{1k}^{(1)} - y_3\delta_{k\alpha}u_{3,\alpha x}^{(0)} + U_k(\mathbf{x}') \tag{4.1}$$

The last term appears because (3.1) is the problem described by variables y_3 and t. In a similar manner [3], it can be verified that, whenever there is a unique solution of the problem of deforming a plate with the governing relations (3.7), this solution is zero, i.e. $u_\alpha^{(0)} = 0$ ($\alpha = 1, 2$). As a result, we have

$$u_\alpha^{(1)} = -y_3u_{3,\alpha x}^{(0)} + U_\alpha(\mathbf{x}'), \quad u_3^{(1)} = U_3(\mathbf{x}') \tag{4.2}$$

Substituting expression 4(2) into (2.4) for $m = -2$ we obtain

$$\sigma_{ij}^{(-2)} = a_{ijk3}u_{k,3y}^{(2)} - y_3a_{ij\beta\alpha}u_{3,\alpha x\beta x}^{(0)} + a_{ijk\alpha}U_{k,\alpha x} + \Gamma_{ijk3}u_{k,3y}^{(2)} - \Gamma_{ij\beta\alpha}y_3u_{3,\alpha x\beta x}^{(0)} + \Gamma_{ijk\alpha}U_{k,\alpha x} \tag{4.3}$$

and (2.5) leads to the problem

$$\sigma_{i3,3y}^{(-2)} = 0, \quad \sigma_{i3}^{(-2)} = 0 \quad \text{for} \quad y_3 = \pm\tfrac{1}{2} \tag{4.4}$$

The solution of problem (4.4) can be represented in the form $\mathbf{u}^{(2)} = \mathbf{u}_1^{(2)} + \mathbf{u}_2^{(2)} + \mathbf{u}_3^{(2)}$, where $\mathbf{u}_1^{(2)}$ and $\mathbf{u}_2^{(2)}$ are similar to the functions $\mathbf{u}_1^{(1)}$ and $\mathbf{u}_3^{(1)}$ introduce above [but $\gamma_{\beta\alpha}$ are calculated using $\mathbf{U}(\mathbf{x}')$ and not $\mathbf{u}^{(0)}(\mathbf{x}')$] and $\mathbf{u}_2^{(2)}$ is the solution of the problem

$$\sigma_{i3,3y}^{u} = 0, \quad \sigma_{i3}^{u} = 0 \quad \text{for} \quad y_3 = \pm\tfrac{1}{2} \tag{4.5}$$

$$\sigma_{i3}^{u} = a_{i3k3}u_{3k,3y}^{(2)} + \Gamma_{i3k3}u_{3k,3y}^{(2)} - y_3a_{ij\beta\alpha}u_{3,\alpha x\beta x}^{(0)} - y_3\Gamma_{ij\beta\alpha}u_{3,\alpha x\beta x}^{(0)} \tag{4.6}$$

The solution of problem (4.5), (4.6) may be found in the form

$$u_{3k,3y}^{(2)} = C_k(t)\,y_3, \quad \text{i.e.} \quad u_{3k}^{(2)} = C_k(t)\frac{y_3^2}{2} + B_k \tag{4.7}$$

In fact, relation (4.6) reduces to $\sigma_{i3}^{u} = 0$. Substituting expression (4.7) into (4.5) we obtain

$$\begin{aligned}\sigma_{i3}^{u} &= a_{i3k3}C_ky_3 + \Gamma_{i3k3}C_ky_3 - y_3a_{i3\beta\alpha}u_{3,\alpha x\beta x}^{(0)} - \\ &- y_3\Gamma_{i3\beta\alpha}u_{3,\alpha x\beta x}^{(0)} = y_3\,\{(a_{i3k3})(\mathrm{C} - (a_{i3k3})^{-1}(a_{i3\beta\alpha}u_{3,\alpha x\beta x}^{(0)}) + \\ &+ (\Gamma_{i3k3})(\mathrm{C} - (\Gamma_{i3k3})^{-1}(\Gamma_{i3\beta\alpha}u_{3,\alpha x\beta x}^{(0)})\} = 0\end{aligned} \tag{4.8}$$

In the general case the operator equation (4.8) (differential, if the material is visco-elastic) has a solution given by operators that are non-local with respect to time, which depend on the curvatures $\{u_{3,\alpha x\beta x}^{(0)}\}$.

From a comparison of Eqs (4.8) and (3.4) it follows that all the conclusions of Sec. 3 hold with respect to (4.8). In particular, when an axial extension is replaced by cylindrical bending in the foregoing example, we obtain the same memory parameters for the plate.

Therefore, in the general case, for plates of constant thickness fabricated from homogeneous materials, the governing equations for the plate and those for the material from which the plate is fabricated are of different types.

The effects discovered also occur in thin rods and in thin-walled structures [5–7].

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